Introduction to Dynamical Systems

Solutions Problem Set 8

Exercise 1. Show that the map $\mathcal{L}$ from Lemma 4.3 in Lecture 8 preserves Hölder continuity as follows: there is $\alpha_* > 0$ such that for all $\alpha \in (0, \alpha_*]$, we have that (here $x, y \in \mathbb{R}^n$) if

$$[g]_\alpha := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} < \infty,$$

then

$$[\mathcal{L}^{-1} g]_\alpha \leq C[g]_\alpha.$$

Hint: take advantage of the explicit formulae for $v_\pm$ in the proof of the lemma.

Solution. Consider the explicit formula for v_+ given by

$$v_+ = \sum_{j \geq 0} A_+^j g_+ \circ \psi^{-j},$$

and notice that therefore, by naming $L_{\psi^{-1}}$ the Lipschitz constant of ψ^{-1}, we have

$$\begin{aligned}
\frac{|v_+(x) - v_+(y)|}{|x - y|^\alpha} &\leq \sum_{j \geq 0} \frac{\left| A_+^j \left(g_+(\psi^{-j}(x)) - g_+(\psi^{-j}(y)) \right) \right|}{|x - y|^\alpha} \\
&\leq \sum_{j \geq 0} \|A_+^j\| [g]_\alpha \frac{\left| \psi^{-j}(x) - \psi^{-j}(y) \right|}{|x - y|^\alpha} \leq [g]_\alpha \sum_{j \geq 0} \|A_+^j\| L_{\psi^{-1}}^{\alpha j}.
\end{aligned}$$

Now, $\|A_+^j\| = (\|A_+^j\|^{1/j})^j$, which behaves like $\rho_{A_+}^j$ (the spectral radius of A_+) for large j. Since $\rho_{A_+} < 1$, we may let α be small enough such that $\|A_+^j\|^{1/j} L_{\psi^{-1}}^\alpha < 1$ for large j, meaning that the above is a geometric series.

By replicating this argument for v_-, we conclude that

$$[\mathcal{L}^{-1} g]_\alpha \leq [v_+]_\alpha + [v_-]_\alpha \leq C[g]_\alpha.$$

□

Exercise 2. Show that the function h effecting the conjugation in Proposition 4.2 is Hölder continuous (for a sufficiently small Hölder exponent).

Solution. We begin by denoting

$$T(\hat{h}) = L^{-1}(\hat{\psi} - \hat{\phi} \circ h).$$

Then, Proposition 4.2 establishes that

$$\lim_{j \to \infty} T^j(0) = \hat{h}_*$$

is the fixed point in the L^∞-topology. It suffices to show that there is a uniform constant D such that

$$\left| T^j(0)(x) - T^j(0)(y) \right| \leq D|x - y|^\alpha, \quad \forall j \geq 1.$$

for a small enough $\alpha > 0$. Then we have (with $h_j = T^j(0)$)

$$|\hat{h}_*(x) - \hat{h}_*(y)| = \lim_{j\to\infty} |h_j(x) - h_j(y)| \le D \cdot |x-y|^\alpha.$$

To show the uniform Holder continuity for the h_j, we argue by induction. This bound is true for $h_1(0) = L^{-1}(\hat{\psi} - \hat{\phi}(0))$. For $|x-y| \ge 1$ use the boundedness of the function. Now assume it holds true for some $j \ge 1$. Then for $\alpha > 0$ small enough, **Exercise 1** guarantees that

$$[T(\hat{h})(x) - T(\hat{h}(y)]_\alpha \lesssim [\hat{\psi} - \hat{\phi} \circ h]_\alpha,$$

and therefore we can estimate

$$[\hat{\psi}]_\alpha \le \sup_{x\neq y, |x-y|\le 1} \frac{|\hat{\psi}(x) - \hat{\psi}(y)|}{|x-y|^\alpha} + 2\|\hat{\psi}\|_{L^\infty} \le D_1.$$

We also get

$$[\hat{\phi} \circ h]_\alpha \le \|\hat{\phi}\|_{C^{0,1}} \cdot [x+h]_\alpha + 2\|\hat{\phi}\|_{L^\infty} \le \epsilon \cdot [h]_\alpha + D_2,$$

where the term D_2 accounts for the situation when $|x-y| \ge 1$. It follows that if $[\hat{h}]_\alpha \le D$, then we have

$$[T(\hat{h})]_\alpha \le D_1 + \epsilon \cdot [h]_\alpha + D_2 \le D_1 + D_2 + \epsilon \cdot D \le D$$

provided that $D = D(D_1, 2, \epsilon)$ is large enough. We conclude by induction that $T^j(0)$ satisfies uniform Holder bounds, which then gives the conclusion after taking the limit. $\square$